

An Analytical Model of Anisotropic Star

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Abstract

In this work a new class of relativistic solutions for anisotropic strange star has been investigated taking into account the inner spacetime represented by Durgapal IV metric potential. We have obtained the solutions of the Einstein field equations in exact form to construct a viable stellar model. The basic characteristics parameter relevant to model are well obeyed in the present scenario. In the present model, the maximum mass is determined by maximizing the radial sound velocity at the centre. We note that, for an isotropic stellar configuration, the maximum masses are found $1.911 M_{\odot}$, $2.123 M_{\odot}$ and $2.335 M_{\odot}$ respectively for radii $b = 9$ Km, $b = 10$ Km and $b = 11$ Km. It is observed that in presence of anisotropy maximum mass increases. Using our model, the radii of a few recently spotted compact stars and pulsars have been predicted and we note that our theoretical predictions align well with the estimated results from observations. Moreover, all of the energy requirements and stability standards needed to define a stable stellar structure are achieved by the present model.

Keywords: GTR, compact objects, Einstein field equation , Equation of state

1 Introduction

since few decades the study of astrophysical phenomena in General Relativity (GR) has attracted the attention of researchers. In various periods, numerous astrophysical models have been devised for this aim. Compact stars are among the most fascinating topics in this regime. The relativistic aspect of astrophysics focuses mostly on the end state of stellar evolutionary cycle. These remnants of supernovae or planetary

nebulae are the constituents members of compact objects (CO) family. There are different categories of COs *viz.*, White Dwarf (WD), Neutron Star (NS), Strange Stars (SS), Black Hole (BH) and Gravastars. Studies showed that these remnants have high density and small radius. Several studies have proposed that some exotic kind of matter may be present adjacent to the core region of NS. The compact object NS may be consisted of interior density of the order of $\sim 10^{15} \text{ gm/cm}^3$ as it has some exotic kind of matter near the core region. Itoh [1] proposed the existence of quarks in the high density core region of NS in hydrostatic equilibrium. Moreover, Bodmar [2] and Witten [3] enlightened a new class of compact objects which established the existence of quark core having suitable stable stellar structure. These are termed as Strange Stars (SS). The most valuable aspect for the determination of physical behavior of NS or SS is equation of state (EoS) which is basically the relation between pressure(p) and density(ρ). In the theoretical investigations of SS, the MIT bag model EoS [4–8] is given by the relation $p = \frac{1}{3}(\rho - 4B)$ where, B is bag constant. This well known EoS demonstrates the stellar configuration that can predict the observational results.

previously, significant stellar models that assumed isotropic pressure inside the stellar configuration were constructed and they discovered numerous behavioral aspects of compact objects. Later, this concept was modified as stellar models constructed based on isotropic pressure concept were unable to explain the basic characteristic features of compact objects. Moreover, it was found that considering anisotropic matter inside the stellar structure one can obtain more realistic stellar models. Investigators found that the consideration of pressure anisotropy inside a highly dense stellar object would play a vital role in the observed properties. A number of works has been obtained considering pressure anisotropy. Investigations suggest that there are many reasons of occurrence of anisotropy in the constituent matter. The presence of anisotropy inside a compact object in extremely dense regimes ($\sim 10^{15} \text{ gm/cm}^3$) was first confirmed by Ruderman [9] and Canuto [10]. The studies of Heintzmann and Hillebrandt [11] found that the presence of anisotropy might play an important role in the red-shift function and the ratio of mass-radius in neutron star (NS). They showed that because of large value of anisotropy factor there is no definitive limit of compactness (mass-radius ratio). The possible origin of pressure anisotropy was studied in details by Herrera and Santos[12]. The investigation of Kippenhahn and A. Weigert [13] indicates that the existence of superfluid of type 3A in the core region leads to originate a anisotropic compact object. Moreover, the Super conductivity and superfluidity behavior may be explained by the presence of pressure anisotropy inside a star as predicted in ref.[14]. The study of Sawyer [15] suggests that pion condensation may also give rise the occurrence of pressure anisotropy inside a compact object. There are few more possible reason of arising pressure anisotropy inside a compact object. The presence of solid core and phase transition are also responsible for the occurrence of pressure anisotropy as predicted by [16]. Many investigators [17–19] have developed stellar models assuming anisotropic matter inside the stellar structure and found different behavioral aspect of compact objects. The study of Bowers [14] suggests that pressure anisotropy may play crucial role on the red-shift function and presents a stellar configuration that is more accurate than the isotropic one. The presence of pressure anisotropy may affect

the physical properties and structure of the stellar configuration [20]. Additionally, anisotropy might influence the maximum mass and radius of a star [21–25].

The principle motivation of this work is to build up adequate stellar model by obtaining the solution of Einstein Filed equation and explain the properties compact objects. The model is then used to explain the various properties specially having high mass and their physical characteristics like maximum mass-radius relation, energy density (ρ), radial (p_r) and transverse (p_t) pressures as well as anisotropy in pressure (Δ), compactness, surface red-shift (Z_s) etc. in the framework of Durgapal IV type metric potential. It is interesting to note that radius of many pulsars can be predicted with great accuracy and the results are consistent with the estimated value of radius from observation. The design of this document is as follows: in the Sect. 2, we obtain the solution of EFEs assuming anisotropic matter inside the star and the criterion for obtaining viable stellar model. In Sect. 3, we discuss different bounds of the model. 4 enlightens the quantities viz mass, red shift and compactness function variations. Numerical analysis are discussed in Sect. 5. Physical analysis of the model are shown in Sect. 6. In Sect. 7 we discuss the energy conditions. Different stability conditions are studied in Sect. 8. Application of the model comprises in Sect. 9. The result and concluding remarks are shown in Sect. 10.

2 Einstein field equation and their solution

In this section, we have explored the solution of EFE and obtain viable stellar model. For a spherically symmetric cold compact star, the interior space time metric is presented as:

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

here $\nu(r)$ and $\lambda(r)$ are two radial co-ordinate dependent unknown metric potentials. These two quantities are responsible for construction of the stellar model. The quantity that changes the space-time curvature known as energy momentum tensor T_{ij} which can be written as,

$$T_{ij} = \text{diag} (-\rho, p_r, p_t, p_t), \quad (2)$$

where, the quantities ρ , p_r and p_t are respectively indicate energy density, radial and transverse pressures. The anisotropy in pressure (Δ) is defined as $\Delta = (p_t - p_r)$. The Einstein field equation (EFE) is given by,

$$\mathbf{R}_{ij} - \frac{1}{2}g_{ij}\mathbf{R} = k^2 T_{ij}, \quad (3)$$

where, the Ricci tensor and Ricci scalar are represented by $\mathbf{R}_{ij}$ and $\mathbf{R}$ respectively. T_{ij} indicates the energy momentum tensor of the interior matter distribution. The factor $k^2 = \frac{8\pi G}{c^2}$, where G is known as Newtonian constant. c is the velocity of light. In astrophysical units, we consider $G = 1$ and $c = 1$. ρ , p_r and p_t are the physical quantities of interest and can be found from the solution of EFEs. Using Eqs. (1) and (2),

Eq. (3) results the following subsequent collection of equations:

$$e^{-\lambda} \left(\frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = k^2 \rho, \quad (4)$$

$$e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = k^2 p_r, \quad (5)$$

$$e^{-\lambda} \left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda' \nu'}{4} - \frac{(\lambda' - \nu')}{2r} \right] = k^2 p_t. \quad (6)$$

We have obtained the following equation using the relation, $\Delta = (p_t - p_r)$ and the expressions of p_r and p_t are taken from Eqs. (5) and (6) respectively:

$$e^{-\lambda} \left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda' \nu'}{4} - \frac{(\lambda' + \nu')}{2r} - \frac{1}{r^2} \right] + \frac{1}{r^2} = k^2 \Delta, \quad (7)$$

where, the overhead-dash indicates the differentiation with respect to r .

Relativistic stellar configuration must obey the following conditions as mentioned below:

- The solution must be free from any kind of singularities. The regularity of the metric potentials suggests that $e^{-\lambda}$ and e^ν must be non zero and positive through out the star. The central density, central pressure are well defined and regular from center to surface.
- The solution should provide pressure and energy density values that are both positive and monotonically decreasing with respect to r . This implies that the pressure and density gradient should be $(\frac{dp_r}{dr}) < 0$, $(\frac{dp_t}{dr}) < 0$ and $(\frac{d\rho}{dr}) < 0$
- At $r = 0$, the anisotropy in pressure, $\Delta = 0$ [14]. It is also necessary that Δ is regular from center to surface of the compact star.
- The surface red-shift (Z_s) should be positive, finite at the stellar boundary.
- Energy conditions related to the anisotropic fluid sphere should be satisfied.
- The causality conditions hold good i.e $0 \leq (\frac{dp_r}{d\rho}) \leq 1$ and $0 \leq (\frac{dp_t}{d\rho}) \leq 1$ inside the star.
- For a stable anisotropic compact star, $0 \leq |v_t^2 - v_r^2| \leq 1$ must be satisfied.
- The maximum compactness bound, i.e., the ratio of mass to radius, should obey the Buchdahl limit [26], viz., $\frac{M}{b} < \frac{4}{9}$.
- The adiabatic index is defined as $\Gamma = \left(\frac{\rho + p_r}{p_r} \right) \frac{dp_r}{d\rho}$. Its value must be $\Gamma \geq \frac{4}{3}$ (NL). However, due to the presence of pressure anisotropy, the limit of Γ is modified and expressed as $\Gamma'_{max} = \frac{4}{3} + \left[\frac{4}{3} \frac{p_t - p_r}{r|p_r|} \right]_{max}$. For a stable stellar model, $\Gamma > \Gamma'_{max}$ must be ensured.

The construction of stellar model consists of finding the unknown quantities relevant to the model. In the present model, if the values of ν , λ , p_r , p_t and ρ are obtained then one can established the stellar model. For this purpose, we first need to chose the metric potentials i.e the values of ν and λ . There are a number of form of metric potentials are available and many authors used them to find stellar model in different time. In the present scenario, we adopt the g_{tt} component of Durgapal *IV* metric potentials [27] as $e^\nu = H(1+x)^4$, where $x = hr^2$ and H is a constant. Using this g_{tt} component and Eqs. (5), (6) and (7) and with help of anisotropy expression $\Delta = (p_t - p_r)$, we then determine the value of g_{rr} component of metric potential.

Using the following transformation as given by Durgapal $e^{-\lambda} = Y(x)$ and $x = hr^2$, where h is an another constant, we rewrite the Eq. (7) as

$$\frac{dY}{dx} + \frac{(8x^2 - (1+x)^2)}{x(x+1)(1+5x)}Y + \frac{(1+x)}{(1+5x)} \left[\frac{1}{x} - \frac{\Delta}{h} \right] = 0, \quad (8)$$

Eq. (8) is an first order differential equation and its solution may be obtained by considering suitable form of pressure anistropy Δ . The basis characteristics of choosing pressure anisotropy is that it should be regular and well behaved through out the interior of the stellar structure. There are many form of pressure anisotropy available [18, 28–31] which provides viable results of stellar model. In this model , we assume the pressure anisotropy as given by;

$$\frac{\Delta}{h} = \frac{\kappa x(3+x)}{(1+x)^3}, \quad (9)$$

where, κ is anisotropy multiplier whose values are taken arbitrarily so that one can construct viable model. We notice that this form of Δ is well behaved and regular inside the stellar structure. Using the value of Δ stated in Eq. 9, we obtain the solution of differential Eq. (8) as ;

$$e^{-\lambda} = \frac{1}{(1+x)^2} \left[\frac{7 + 10x(\kappa - 1) + x^2(\kappa - 1)}{7} \right] + \frac{Ax(1+5x)^{-2/5}}{(1+x)^2}, \quad (10)$$

where, the value of the constant A can be determined by analyzing the boundary conditions. For $\kappa = 0$, it is evident that Eq. (10) simplifies to the Durgapal *IV* isotropic solution [27]. Now we are able to determine the relevant quantities namely density ρ , radial pressure p_r , transverse pressure p_r , gravitational mass M , compactness u and surface red-shift z_s using the vales of ν and λ . According to Eqs. (4) and (5), the formulas for matter density and radial pressure are generated as,

$$\rho = \frac{h}{(1+x)^2} \left[\frac{4Ax}{(1+5x)^{7/5}} - 2\Psi_1 \right] + \frac{h(3x-1)\Psi_2}{x(1+x)^3} + \frac{h}{x}, \quad (11)$$

$$p_r = -\frac{h}{x} + \frac{h\Psi_2(1+9x)}{x(1+x)^3}, \quad (12)$$

and the transverse pressure

$$p_t = \Delta + p_r, \quad (13)$$

where we use,

$$\begin{aligned}\Psi_1 &= \frac{A}{(1+5x)^{\frac{2}{5}}} + \frac{10(\kappa-1) + 2x(\kappa-1)}{7} \\ \Psi_2 &= \frac{Ax}{(1+5x)^{\frac{2}{5}}} + \frac{7 + 10x(\kappa-1) + x^2(\kappa-1)}{7}.\end{aligned}\quad (14)$$

The boundary conditions yield the constants h , H , and A . At the boundary i.e ($r = b$) of the fluid sphere, we note the following:

- At the boundary of the stellar structure, we match the interior solution with the exterior schwarzschild [32] vacuum solution and found that

$$e^{\nu(r=b)} = e^{-\lambda(r=b)} = \left(1 - \frac{2M}{b}\right). \quad (15)$$

Where, M is the gravitational mass contained within the sphere of radius b . Hence the values of e^ν and $e^{-\lambda}$ at the boundary become

$$\begin{aligned}\frac{1}{(1+X)^2} \left[\frac{7 + 10X(\kappa-1) + X^2(\kappa-1)}{7} \right] \\ + \frac{AX(1+5X)^{-2/5}}{(1+X)^2} = \left(1 - \frac{2M}{b}\right),\end{aligned}\quad (16)$$

and

$$H(1+X)^4 = \left(1 - \frac{2M}{b}\right), \quad (17)$$

where $X = hb^2$.

- The radial pressure which is monotonically decreasing function of radial distance from the center of the sphere vanishes at the surface. This means that at $r = b$,

$$p_r(r = b) = 0. \quad (18)$$

Now one can determine the values of model parameter A , h and H using Eqs. (16), (17) and (18) and hence the viable stellar model.

3 Different bounds of the model

The value of model parameter A may be obtained from Eq. (18) which is given as:

$$A = -\frac{(32 + 10\kappa + X^2(9\kappa - 16) + 7X(13\kappa - 16))}{7(1+5X)^{-\frac{2}{5}}(1+9X)}, \quad (19)$$

and again equating the values of $e^{\nu(r=b)}$ and $e^{-\lambda(r=b)}$, we can determine the value of H as:

$$H = \left[\frac{7 + 10X(\kappa - 1) + X^2(\kappa - 1)}{7(1 + X)^6} + \frac{AX(1 + 5X)^{-\frac{2}{5}}}{(1 + X)^6} \right]. \quad (20)$$

Therefore, we obtain the expressions of mass using Eq. (20) and the value of e^ν at $r = b$ as:

$$M = \frac{b}{2} (1 - H(1 + hb^2)^4). \quad (21)$$

Knowing the value of H and h , we can easily determine the mass of a star.

3.1 Central density, central pressure and their ratio

The metric potentials e^λ and e^ν picks up finite value at $r = 0$ such as $e^{\lambda(r)} = 1$ and $e^{\nu(r)} = H$. However, the first derivative of these metric potentials are equal to zero at $r = 0$, i.e. $(e^{\lambda(r)})'_{r=0}$ and $(e^{\nu(r)})'_{r=0}$. This indicates that e^λ and e^ν are well behaved as well as regular throughout the star. The values of central density and pressure in terms of metric parameters can be calculated from Eqs. (16) and (17) at $r = 0$ which are given as:

$$\rho(0) = \left(\frac{72 - 21A - 30\kappa}{7} \right) h, \quad (22)$$

$$p_r(0) = \left(\frac{32 + 7A + 10\zeta}{7} \right) h, \quad (23)$$

and

$$\left(\frac{p_r}{\rho} \right)_{r=0} = \frac{32 + 7A + 10\kappa}{72 - 21A - 30\kappa}. \quad (24)$$

Using the inequalities $\rho(0) > 0$, $p_r(0) > 0$ and $3p_0 + \rho_0 > 0$, we note that $h > 0$. It is noticeable from Eqs. (27) and (28) that the anisotropic parameter κ affects both the central density and the radial pressure.

4 Mass, compactness and red-shift functions

In this section, we determine the total gravitational mass inside of a stellar structure of radius b . The equation that can be used to calculate the mass is given by,

$$M(b) = 4\pi \int_0^b r^2 \rho(r) dr, \quad (25)$$

where $\rho(r)$ is matter density. Putting $x = hr^2$ and integrating Eq. (25), we have calculated the mass expression as:

$$M(b) = \frac{hb^3}{14(1 + hb^2)^2} \left[24 + (8 - \kappa)hb^2 - 10\kappa - \frac{7A}{(1 + 5hb^2)^{2/5}} \right]. \quad (26)$$

The corresponding compactness and red-shift functions are given as:

$$u(b) = \frac{hb^2}{14(1+hb^2)^2} \left[24 + (8-\kappa)hb^2 - 10\kappa - \frac{7A}{(1+5hb^2)^{2/5}} \right], \quad (27)$$

and

$$z_s(b) = \left(\frac{1}{\sqrt{1-2u(b)}} - 1 \right). \quad (28)$$

From Eq. (26), it is evident that as $b \rightarrow 0$, $M(b) \rightarrow 0$. This indicates that the mass function in this model is regular. Fig. (1) shows the radial variation of mass. The profile of compactness (u) and red-shift function (z_s) are shown in figs. (2) and (3). We have identified a non-zero maximum red-shift at the surface of the star.

5 Numerical analysis of the model

In this part, we implemented a numerical approach to determine the maximum mass of the star. For this purpose, we take radius as input parameter and then applying suitable boundary conditions we determine different model parameters and hence the maximum mass of the star. First, we assume radius b as input parameter and then we solve eq. (18) and maximizing $\frac{dp_r}{d\rho}$ at the center. We apply same methods for different chooses of anisotropy parameter κ . Using the values of A , H and h the maximum mass has been calculated from $M_{max} = \frac{b}{2}(1 - e_b^{-\lambda})$, the value of $e^{-\lambda}$ at $r = b$ is represented by $e_b^{-\lambda}$.

5.1 Model-I

Here, we consider star having radius $b = 9 \text{ Km}$. Applying the numerical procedure stated above we determine the values of model parameters A , H and h . Using these values, we then calculate the mass of the star. In tab. (1) we have shown the calculated maximum mass for different values of κ . We note that for $\kappa = 0$ i.e for an isotropic star the maximum mass is found $M_{max} = 1.911 M_\odot$. We notice that maximum mass increases with increasing value of κ . The compactness is found as $u = 0.3131$.

5.2 Model-II

Here, we consider star having radius $b = 10 \text{ Km}$. Applying same procedure we determine the values of model parameters A , H and h . Using these values, we then calculate the mass of the star. In tab. (2) we have shown the calculated maximum mass for different values of κ . We note that for $\kappa = 0$ i.e for an isotropic star the maximum mass is found $M_{max} = 2.123 M_\odot$. We notice that maximum mass increases with increasing value of κ . The compactness is found as $u = 0.3131$.

5.3 Model-III

Here, we consider star having radius $b = 11 \text{ Km}$. Applying same procedure we determine the values of model parameters A , H and h . Using these values, we then calculate the mass of the star. In tab. (3) we have shown the calculated maximum mass for different values of κ . We note that for $\kappa = 0$ i.e for an isotropic star the maximum mass is found $M_{max} = 2.335 M_{\odot}$. We notice that maximum mass increases with increasing value of κ . The compactness is found as $u = 0.3131$.

Table 1 Maximum mass $M_{max} (M_{\odot})$ for model-I of radius taken $b = 9 \text{ Km}$ for different values of κ .

κ	A	H	$h (km^{-2})$	$M_{max}(M_{\odot})$	u_{mx}	Z_{smax}
0.00	-0.0714286	0.146006	0.00327047	1.911	0.3131	0.6357
0.30	-0.16250	0.095468	0.00439311	2.067	0.3387	0.7606
0.50	-0.223214	0.0654966	0.00552891	2.173	0.3561	0.8640

Table 2 Maximum mass $M_{max} (M_{\odot})$ for model-II of radius taken $b = 10 \text{ Km}$ for different values of κ .

κ	A	H	$h (km^{-2})$	$M_{max}(M_{\odot})$	u_{mx}	Z_{smax}
0.00	-0.0714286	0.146006	0.00264908	2.123	0.3131	0.6357
0.30	-0.16250	0.095468	0.00355842	2.296	0.3387	0.7606
0.50	-0.223214	0.0654966	0.00447842	2.414	0.3561	0.8640

Table 3 Maximum mass $M_{max} (M_{\odot})$ for model-III of radius taken $b = 11 \text{ Km}$ for different values of κ .

κ	A	H	$h (km^{-2})$	$M_{max}(M_{\odot})$	u_{mx}	Z_{smax}
0.00	-0.0714286	0.146006	0.00218932	2.335	0.3131	0.6357
0.30	-0.162500	0.095468	0.00294084	2.523	0.3387	0.7606
0.50	-0.223214	0.065497	0.00370177	2.923	0.3561	0.8640

6 Physical analysis of the model

We have analyzed the different physical parameters of the star such as metric potentials ($e^{-\lambda}$, e^{ν}), radial pressure (p_r), transverse pressure (p_t), Δ etc. We have randomly chosen κ so that the radial pressure at the surface is zero and the stellar model behaves

Table 4 Maximum radius b_{max} (km) and mass M_{max} ($M_\odot$) as predicted from the model and TOV approach for different values of κ .

	Data from the model			Data from TOV approach		
	κ	Radius (b)	$M_{max}(M_\odot)$	κ	Radius (b_{max})	$M_{max}(M_\odot)$
Model-I	0	9	1.91	0	8.67	2.05
	0.3	9	2.07			
	0.5	9	2.17			
Model-II	0	10	2.12	0	9.87	2.29
	0.3	10	2.30			
	0.5	10	2.41			
Model-III	0	11	2.34	0	10.71	2.53
	0.3	11	2.52			
	0.5	11	2.66			

well. Figs. (1), (2), and (3) illustrate how the red shift function, compactness, and gravitational mass changes in relation to radial distance. The profiles of e^ν and $e^{-\lambda}$ with radial distance are shown in fig. (4). From the nature, we note that at $r = 0$, $e^\lambda = 1$ while at the surface $e^{-\lambda} = e^\nu$. Additionally, $e^{-\lambda}$ reaches a minimal value at the surface. For various values of κ , the relationship between radial distance and energy density (ρ), radial pressure (p_r), transverse pressure (p_t), and pressure anisotropy (Δ) has been examined. Figure. (6) illustrates how the energy density changes as r . It is observed that the density is greater near the center and falls outward radially. This suggests at the star's interior has a negative density gradient. In fig. 7, the radial pressure (p_r) fluctuation is displayed while fig. (8) displays the transverse pressure (p_t). Like density gradient pressure gradients are also negative inside the star. Figure. (9) is the variation of anisotropy in pressure with radial distance. This figure shows that Δ is zero at the center and increases towards the surface. It attains maximum value at the surface. Hence the anisotropy in pressure is regular inside the star and well behaved.

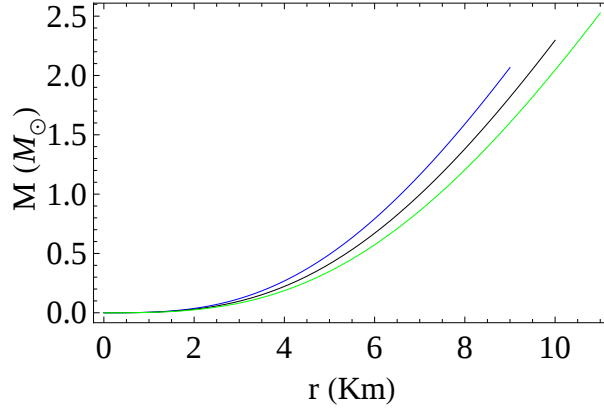


Fig. 1 Profile of $M(M_\odot)$ vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

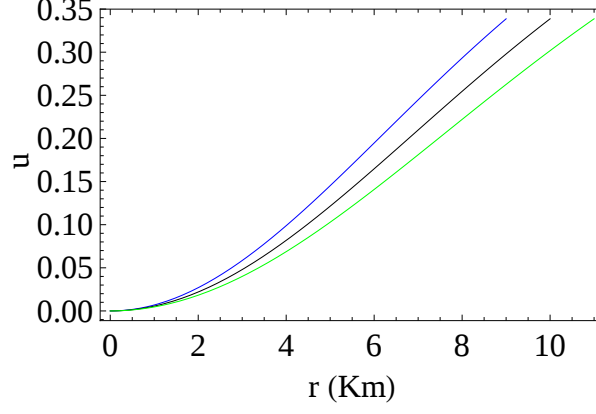


Fig. 2 Profile of u vs r . Blue, Black and green line represent respectively for radii radii 9, 10 and 11 Km at $\kappa = 0.3$.

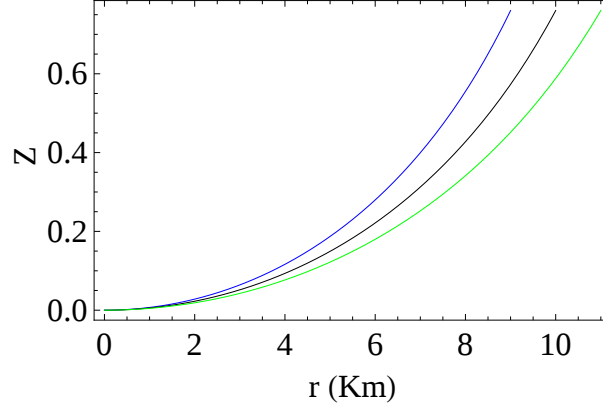


Fig. 3 Profile of Z_s vs b . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

7 Different energy conditions

The satisfaction of energy conditions is a crucial requirement for a physically viable model in the case of an isotropic or anisotropic fluid sphere [33, 34]. For a spherical model to be considered viable, the following energy criterion must be fulfilled :

1. NEC: $\rho + p_r \geq 0, \rho \geq 0$
2. WEC: $\rho + p_r \geq 0, \rho \geq 0, \rho + p_t \geq 0$
3. SEC: $\rho + p_r \geq 0, \rho + p_t \geq 0, \rho + p_r + 2p_t \geq 0$
4. DEC: $\rho \geq 0, \rho - p_r \geq 0, \rho - p_t \geq 0$

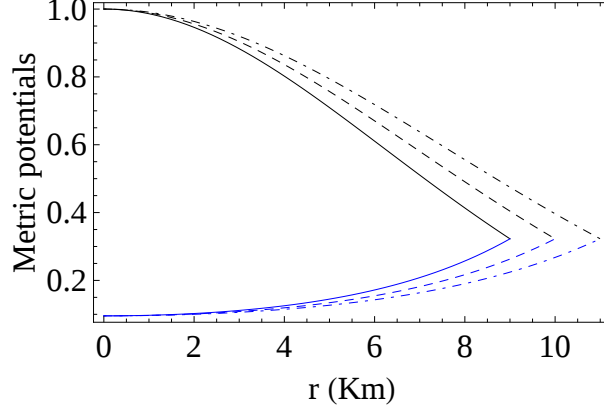


Fig. 4 Profile of $e^{-\lambda}$ vs r and e^{ν} vs r . Solid, dashed and dot dashed line represent $b = 9 \text{ Km}$, $b = 10 \text{ Km}$ and $b = 11 \text{ Km}$ respectively. Blue and black lines represent e^{ν} and $e^{-\lambda}$ at $\kappa = 0.3$.

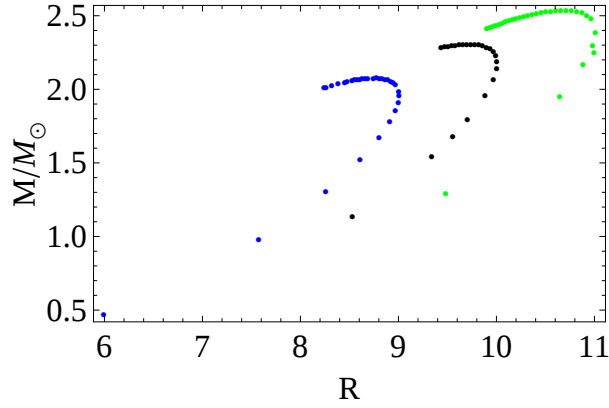


Fig. 5 Profile of $M - R$. Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

Due to complexities of the equations, we have shown all the energy conditions graphically. We have adopted the method as proposed in [34]. In figs. (12) - (15), we have shown all the energy conditions graphically and it is evident that the present model satisfies the required energy conditions.

8 Different stability conditions

To ensure a stable and physically acceptable compact star model, we have studied the following stability criteria in the present scenario:

1. TOV condition
2. Herrera cracking condition,
3. Adiabatic index, and
4. Stability against Lagrangian perturbation.

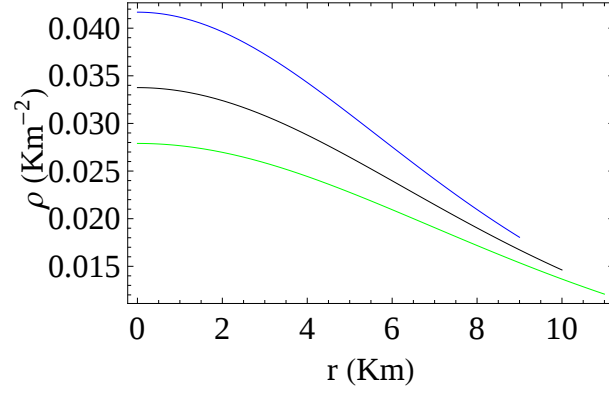


Fig. 6 Profile of ρ vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

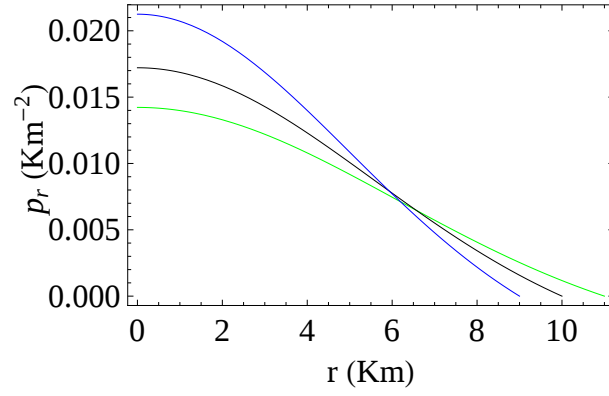


Fig. 7 Profile of p_r vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

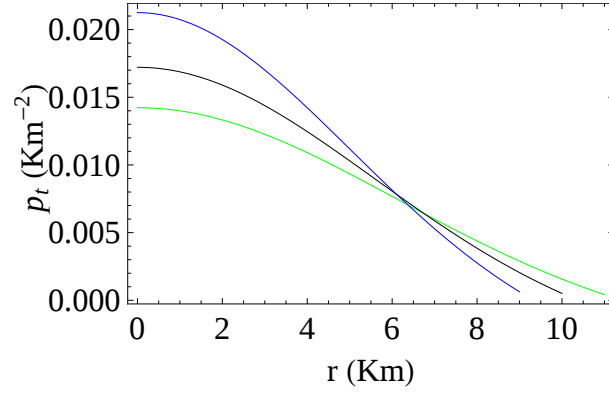


Fig. 8 Profile of p_t vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

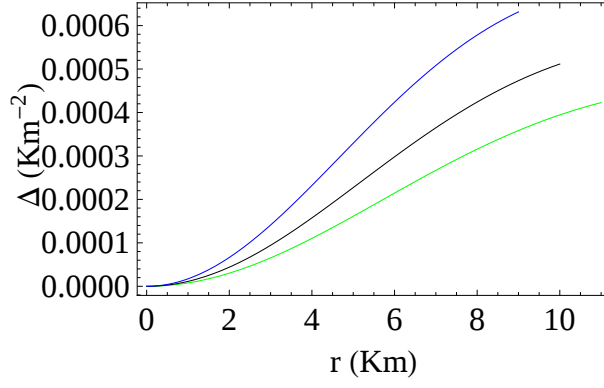


Fig. 9 Profile of Δ vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

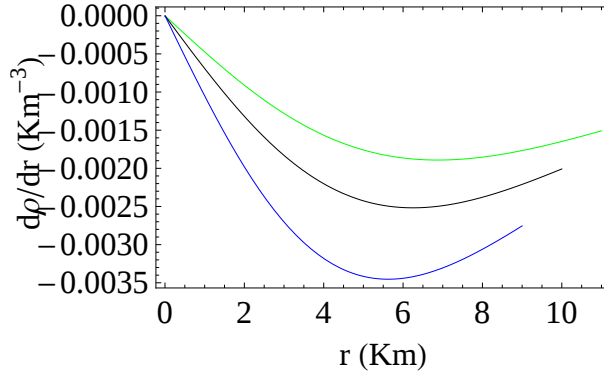


Fig. 10 Profile of p_t vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

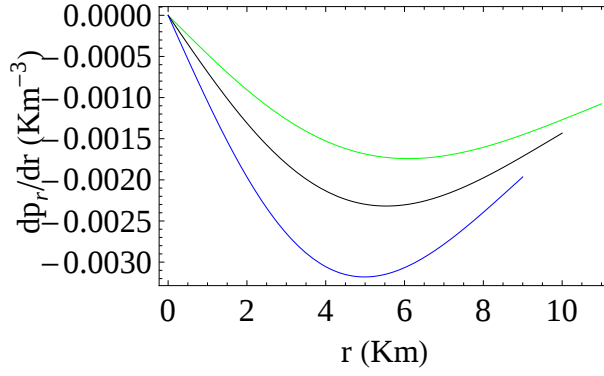


Fig. 11 Profile of Δ vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

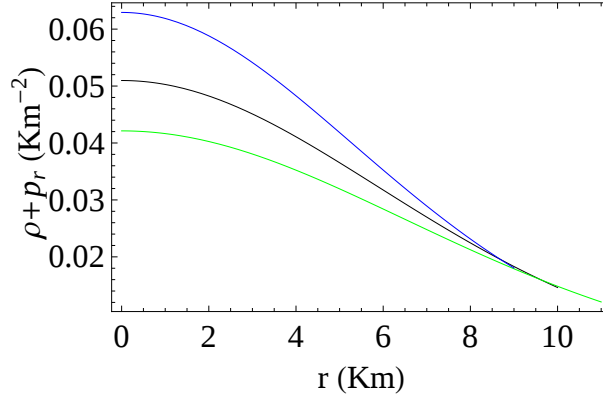


Fig. 12 Profile of $(\rho + p_r)$ vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

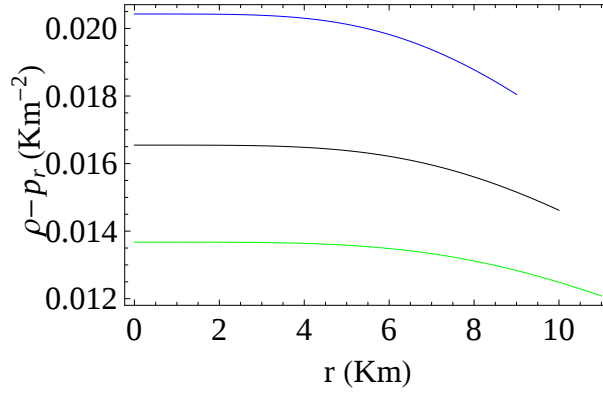


Fig. 13 Profile of $(\rho - p_t)$ vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

8.1 Stability under Generalized TOV condition

The generalized TOV equation that shows how the equilibrium of different forces, present inside a star, namely hydrostatic force (F_h), force due to gravity (F_g) and anisotropic force (F_a) make a fluid sphere stable may be written as:

$$-\frac{M_g(r)(\rho + p_r)}{r^2}e^{(\lambda-\nu)/2} - \frac{dp_r}{dr} + \frac{2\Delta}{r} = 0. \quad (29)$$

According to Tolman-Whittakar [35], the term $M_g(r)$ is given as ,

$$M_G(r) = \frac{1}{2}r^2\nu'e^{(\nu-\lambda)/2}, \quad (30)$$

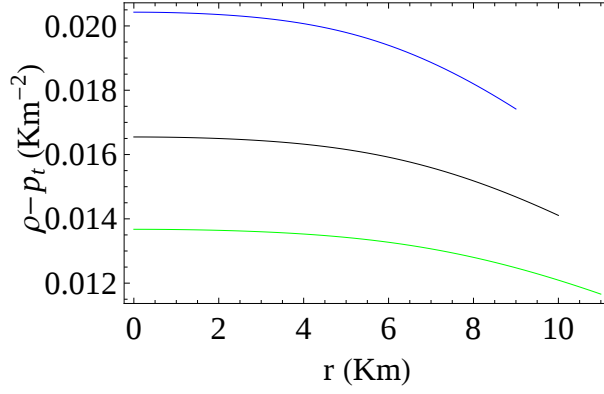


Fig. 14 Profile of $(\rho - p_r)$ vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

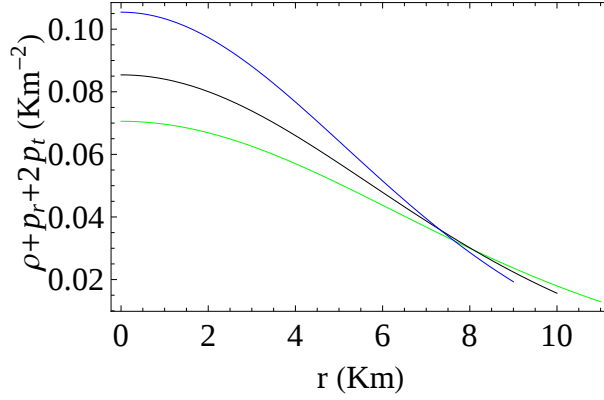


Fig. 15 Profile of $(\rho + p_r + 2p_t)$ vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

using Eq. (30), the TOV equation stated in Eq. (29) may be written as:

$$-\frac{\nu'}{2}(\rho + p_r) - \frac{dp_r}{dr} + \frac{2\Delta}{r} = 0. \quad (31)$$

Here, we define the three forces as $F_g = -\frac{\nu'}{2}(\rho + p_r)$, $F_h = -\frac{dp_r}{dr}$ and $F_a = \frac{2\Delta}{r}$. Using the values of F_g , F_h and F_a , Eq. (31), the eq. (31) reduces to

$$F_g + F_h + F_a = 0. \quad (32)$$

Figure. (18) shows how F_g , F_h , F_a and their resultant vary with respect to radial distance r for three different compact star models and demonstrating that the model clearly satisfies the stability criteria under TOV conditions.

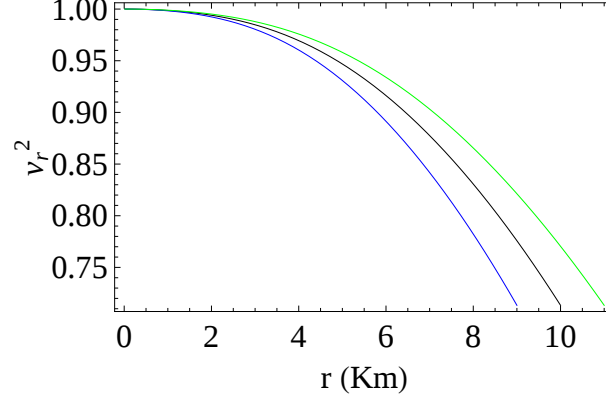


Fig. 16 Profile of v_r^2 vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

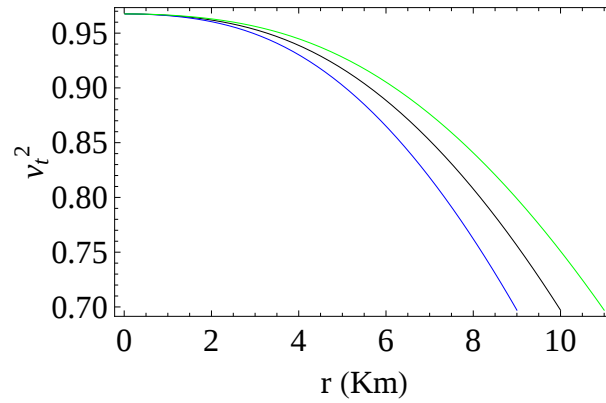


Fig. 17 Profile of v_t^2 vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

8.2 Relativistic Adiabatic index

The adiabatic index Γ of interior matter is an important tool to study the stability of a stellar structure. It represents the resistivity of a gas against compression. In case of a compact star, it demonstrates how rapidly pressure rises in response to an increment of energy density. A sphere with high adiabatic index requires higher compression. In order to obtain a stable stellar configuration, we have to verify the impact of infinitesimal radial adiabatic perturbation. This concept of adiabatic index was first adopted by Chandrasekhar [36]. Subsequently, this concept was taken up by many authors to study the stability of stellar models. The expression for adiabatic index (Γ), in case of isotropic star, can be written as [11]:

$$\Gamma = \frac{(\rho + p_r)}{p_r} \frac{dp_r}{d\rho}. \quad (33)$$

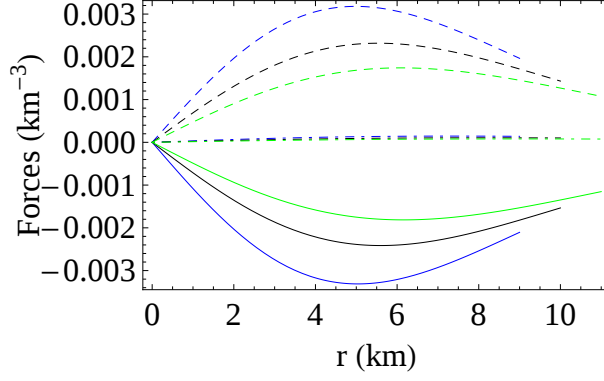


Fig. 18 Radial profile of different force components. Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$. Further, solid, dashed and dot dashed lines indicate F_g , F_h and F_a , respectively.

Now, for a Newtonian ideal fluid sphere the stability under adiabatic index indicates that $\Gamma = \frac{4}{3}$ according to [11]. But this equation becomes more complicated when one consider the anisotropic matter inside the star. For an anisotropic compact stellar object the expression given in Eq. (33) changes to a different form as predicted by Chan et al. [37] as :

$$\Gamma'_{max} = \frac{4}{3} + \left[\frac{4}{3} \frac{(p_t - p_r)}{r|p'_r|} \right]_{max}, \quad (34)$$

In Eq. (34), the term inside bracket represents correction due to anisotropy. As the terms in Eq. (34) are positive, hence for an anisotropic star, Γ will be greater than Γ'_{max} for stability. Fig. 19 displays the adiabatic index fluctuation with radial distance for three distinct star models. It is noted that the value of Γ satisfies the condition imposed by Chan et al. [37]. We therefore believe that our model remains stable even in the presence of adiabatic perturbations.

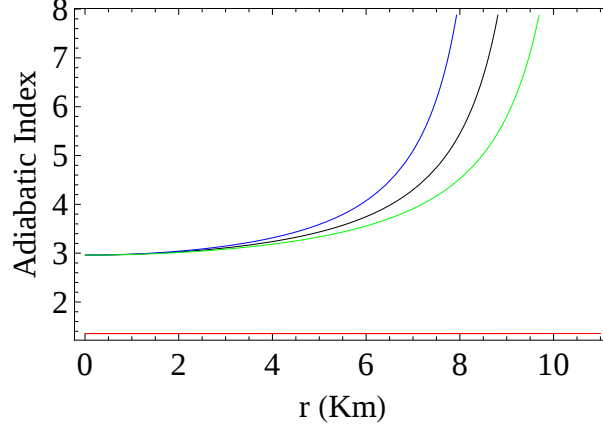


Fig. 19 Profile of Γ vs r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$. Red line indicates $\Gamma = \frac{4}{3}$.

8.3 Herrera Cracking condition

In GR, there is another important criterion called Herrera cracking condition to be satisfied for a stable stellar configuration. The effect of density fluctuation and the local anisotropy on the overall stability of anisotropic fluid distribution may be explored using the concept of Herrera [38]. Such concept was proposed to describe the response of fluid just after the departure from its equilibrium and put forward as an approach to study the stability of a self-gravitating object in an alternative way. Herrera [38] introduced the idea of cracking to inspect if a matter configuration, of anisotropic nature, is stable or not. In accordance with Herrera's idea, Abreu [39] provided a criterion which states that a stellar model will be in stable equilibrium if the radial (v_r) and transverse (v_t) sound speeds meet the requirements given below:

$$0 \leq |v_t^2 - v_r^2| \leq 1. \quad (35)$$

The variation of $|v_t^2 - v_r^2|$ with r for three distinct star models is displayed in fig. 20. Their profiles indicate that the requirement $0 \leq |v_t^2 - v_r^2| \leq 1$ is successfully maintained inside the star.

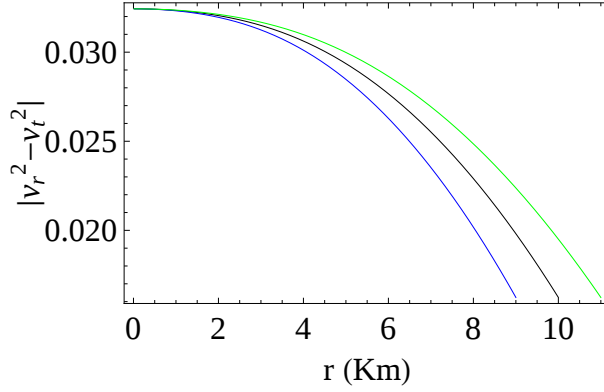


Fig. 20 Profile of $|v_t^2 - v_r^2|$ with r . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

8.4 Stability against Lagrangian perturbation

We evaluate the Lagrangian variation in radial pressure at the stellar surface as a function of the oscillation frequency (ω^2) in order to test the stability of our model with respect to small radial oscillations. Following the methodology introduced by Pretel [40], we investigate the frequency dependence of the Lagrangian perturbation as outlines in Pretel's framework. Consequently, the coupled differential equations that describe the radial oscillations are formulated as follows:

$$\frac{d\eta}{dr} = -\frac{1}{r}(3\eta + \frac{\Delta p_r}{\Gamma p_r}) + \frac{d\nu}{dr}\eta, \quad (36)$$

$$\begin{aligned} \frac{d\Delta p_r}{dr} = & \eta \left(\frac{\omega^2}{c^2} e^{2(\lambda-\nu)} (\rho + p_r) r - 4 \frac{dp_r}{dr} \right. \\ & \left. - \frac{8\pi G}{c^4} (\rho + p_r) e^{2\lambda} r p_r + r(\rho + p_r) \left(\frac{d\nu}{dr} \right) \right) \\ & - \Delta p_r \left(\frac{d\nu}{dr} + \frac{4\pi G}{c^4} (\rho + p_r) r e^{2\lambda} \right). \end{aligned} \quad (37)$$

In this context, η denotes the eigen function associated with the radial component of the Lagrangian displacement and is given by $\eta = \frac{\delta(r)}{r}$. Here, η is normalized so that $\eta(0) = 1$. To prevent a central singularity in Eq. (36), the term containing $(\frac{1}{r})$ must approach zero as $r \rightarrow 0$. Based on this condition, we derive the following criterion:

$$\Delta p_r = -3 \left(\frac{\rho + p_r}{p_r} \frac{dp_r}{d\rho} \right) \eta p_r. \quad (38)$$

Additionally, the change in the radial pressure component, Δp_r , caused by the Lagrangian perturbation's frequency dependence must decrease at the stellar surface, i.e.,

$$\Delta p|_{r=b} = 0. \quad (39)$$

In accordance with fig. (21), ω^2 stays positive for all normal modes of radial

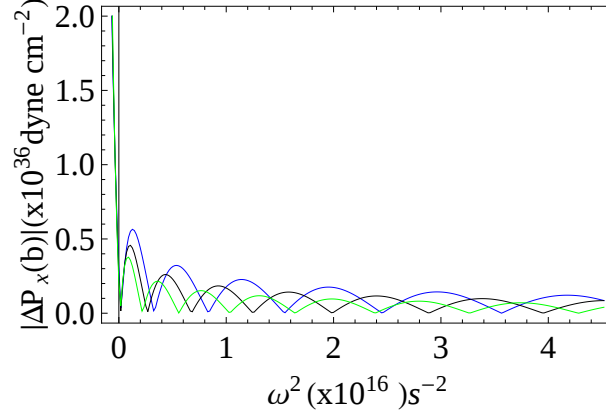


Fig. 21 Profile of Lagrangian perturbation with normal mode frequency ω^2 . Blue, Black and green line represent respectively for radii 9, 10 and 11 Km at $\kappa = 0.3$.

oscillations that are indicated by the plots' minima. This criterion leads us to the conclusion that the model remains stable even when subjected to minor radial oscillatory perturbations.

9 Application of the model

We have predicted the radius of a few recently discovered compact objects using the model. Observational results are in agreement with the values obtained. These results are tabulated in tab 5. Furthermore, in situations where an estimated radius is unavailable, we have predicted the probable radius using our methodology. For example, compact object *PSR J0952 – 0607*, the predicted radius are 11.07 Km for $\kappa = 0$, 10.23 Km for $\kappa = 0.3$ and 9.73 Km for $\kappa = 0.5$. From TOV approach, we noticed that the predicted mass and radius are $2.54 M_\odot$ and 10.85 Km.

10 Discussion

In the present article, we have obtained the singularity free solution of the EFE and determined the physical quantities relevant to the model in the framework of Durgapal IV geometry. Using the g_{tt} component of the metric, we have obtained g_{rr} component in presence of pressure anisotropy. The form of Δ is chosen carefully so that we can obtain a singularity free well behaved exact solution and radial pressure inside becomes positive while zero at the surface. Moreover, the value of anisotropy factor κ is used to parameterized the solution. In the present article, we have determined the maximum masses using numerical procedure stated in sect. 5 for radii $b = 9, 10, 11$ Km. We have obtained $M_{max} = 1.911 M_\odot$, $M_{max} = 2.123 M_\odot$ and $M_{max} = 2.335 M_\odot$ for isotropic case while using TOV approach we notice that the maximum mass that

Table 5 Maximum radius b_{max} (km) and mass M_{max} ($M_\odot$) prediction of recently observed compact objects from the model and TOV approach

	Data from the model			Data from TOV approach	
	κ	Radius (b)	$M_{max}(M_\odot)$	Radius b_{max}	$M_{max}(M_\odot)$
Cen X-3 [41] ($9.178^{+0.13}_{-0.13}$)Km ($1.49^{+0.08}_{-0.08}$) $M_\odot$	0	7.02	1.49	9.19	2.02
	0.3	6.50	1.49		
	0.5	6.17	1.49		
PSR J0952 – 0607 [42] (-) (2.35)	0	11.07	2.35	10.85	2.54
	0.3	10.23	2.35		
	0.5	9.73	2.35		
PSR J0740+6620 [43] (12.39)Km (2.072) $M_\odot$	0	9.76	2.072	12.37	2.73
	0.3	9.02	2.072		
	0.5	8.58	2.072		
GW190814 [44] ($11.76^{+0.14}_{-0.19}$)Km (2.59) $M_\odot$	0	12.20	2.59	11.46	2.702
	0.3	11.28	2.59		
	0.5	10.73	2.59		
Vela X-1 [41] ($9.56^{+0.08}_{-0.08}$)Km ($1.77^{+0.08}_{-0.08}$) $M_\odot$	0	8.34	1.77	9.39	2.12
	0.3	7.71	1.77		
	0.5	7.33	1.77		

$M_{max} = 2.05 M_\odot$ at radius $b_{max} = 8.67 \text{ Km}$, $M_{max} = 2.29 M_\odot$ at radius $b_{max} = 9.87 \text{ Km}$ and $M_{max} = 2.53 M_\odot$ at radius $b_{max} = 10.71 \text{ Km}$. The maximum mass increases with increasing value of κ . Fig. (1) shows the variation of mass with radial distance for three different radii. Similarly, the variations of compactness and red shift function with radial distance has shown in figs. (2) and (3). The compactness profile reveals that at $r = 0$, $u = 0$ and at the surface we note non zero value of u . From fig. (3), it is evident that red-shift function z becomes zero at the center of the star. An well behaved and regular metric potentials as function of radial distance has shown in fig. (4). This profile indicates that $e^{-\lambda}$ becomes unity at $r = 0$ while decreases away from the center and at the stellar surface their numerical values are same. Fig. (5) shows the variation of mass. The model is utilized for determining the maximum mass, compactness, surface red shift, and other parameters and they are then tabulated. (1), (2), and (3) for different choices of κ to make pressure inside the configuration positive. In tab. (4), we have tabulated maximum mass from the model as well as TOV approach. This table shows that data received from the model are in good agreement with data collected from the TOV solution. The present model satisfies the maximum compactness limit i.e Buchdahl limit [26] that defines the flexibility of the model. We noticed that maximum mass increases with increasing value of κ . Figs. (6), (7), (8) and (9) respectively represent density, radial pressure, transverse pressure and pressure anisotropy variation with radial distance. From density profile, we noticed that density is maximum at the center and minimum at surface. Similar variation observed for radial and transverse pressures. Moreover, Radial pressure vanishes at the surface. The profile of density and pressure gradients shown in figs. (10) and (11) indicates that their values are negative throughout the configuration. The validity of

the model also studied by the different energy conditions. Figs. (12)-(15) show the graphical representation of energy conditions. In the present model, we have evaluated the causality conditions presented in figs. (16) and (17), which satisfied the criterion of configuration stability. The radii of some pulsars and compact stars observed recently have been predicted using present model and tabulated in tab 5. The estimated radii of some compact objects may be predicted from the present model and these results are consistent with the results obtained from TOV approach with suitable value κ . Furthermore, when an estimated radius is unavailable, we used our model to determine the probable radius. For example, compact object *PSR J0952 – 0607*, the predicted radius are 11.07 for $\kappa = 0$, 10.23 for $\kappa = 0.3$ and 9.73 for $\kappa = 0.5$. From TOV approach, we noticed that the predicted mass and radius are $2.54 M_{\odot}$ and 10.85 Km . Figure. (18) shows the stability of the model under different forces. The profile of adiabatic index shown in fig. (19) indicates that the value of Γ is satisfied in the present model. The cracking condition proposed by Herrera [38] shown in fig. (20) reveals the validity of $0 \leq |v_t^2 - v_r^2| \leq 1$ throughout the interior of the star. The graphical representation of Lagrangian perturbation is shown in fig. (21). The plot demonstrates that the frequencies of the normal mode of oscillation are real. Hence, the proposed model is stable under this circumstance. Therefore, we may conclude that the present model describes a realistic configuration in presence of anisotropy and useful for the theoretical explanation of estimated results from observations which agrees well with the recent observational evidences.

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